

The `gamlss.family` distributions.

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May 19, 2010

Preface

This booklet shows the probability functions for all distributions available in the **gamlss** package.

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Chapter 1

Distributions in the `gamlss` packages

In this Chapter we provide the mathematical form of the probability function of all `gamlss.family` distributions together with their means and variances.

The distributions in Tables 1.1, 1.2 and 1.3 provide a list of all `gamlss.family` distributions with the default link functions for each of the parameters of the distribution. The tables are constructed according to the type of random variable involve, that is whether the random variable is a *continuous*, a *discrete* or a *mixed* (i.e. a mixture of a continuous and a discrete distribution, for example a continuous distribution with additional point probabilities) random variables. In the rest of the chapter we will put the mixed distributions with the continuous ones (depending of the range possible values of the response) and we will categorize the discrete distributions depending on whether the response variable is a *count* or *binomial type* (i.e. a count out of a *binomial denominator* or total.)

In each case the specific parameterization(s) used by `gamlss` for each of the distributions is given. Note that the `gamlss` package provides, for each parameterization, functions for the probability density function (pdf), cumulative distribution function (cdf), inverse cdf (i.e. quantile) and random number generation. The functions are given by putting each of the letters **d**, **p**, **q** and **r** respectively before the `gamlss.family` name for the particular distribution parameterization. For example, for the parameterization of the normal distribution given by (1.1) below, denoted by $\text{NO}(\mu, \sigma)$, the corresponding `gamlss.family` functions `dNO`, `pNO`, `qNO` and `rNO` define its pdf, cdf, inverse cdf and random number generation respectively. Note also that the package `gamlss.demo` provides visual presentation of all the `gamlss.family` distributions and can be used to examine how changing the parameters effects the shape of the distribution.

1.1 Continuous two parameter distributions on \Re

1.1.1 Normal (or Gaussian) distribution (NO, NO2, NOF)

First parameterization (NO)

The normal distribution is the default of the argument `family` of the function `gamlss()`. The parameterization used for the normal (or Gaussian) probability density function (pdf), denoted

Distributions	R Name	μ	σ	ν	τ
beta	BE()	logit	logit	-	-
Box-Cox Cole and Green	BCCG()	identity	log	identity	-
Box-Cox power exponential	BCPE()	identity	log	identity	log
Box-Cox t	BCT()	identity	log	identity	log
exponential	EXP()	log	-	-	-
exponential Gaussian	exGAUS()	identity	log	log	-
exponential gen. beta type 2	EGB2()	identity	identity	log	log
gamma	GA()	log	log	-	-
generalized beta type 1	GB1()	logit	logit	log	log
generalized beta type 2	GB2()	log	identity	log	log
generalized gamma	GG()	log	log	identity	-
generalized inverse Gaussian	GIG()	log	log	identity	-
generalized t	GT()	identity	log	log	log
Gumbel	GU()	identity	log	-	-
inverse Gaussian	IG()	log	log	-	-
Johnson's SU (μ the mean)	JSU()	identity	log	identity	log
Johnson's original SU	JSUo()	identity	log	identity	log
logistic	LO()	identity	log	-	-
log normal	LOGNO()	log	log	-	-
log normal (Box-Cox)	LNO()	log	log	fixed	-
NET	NET()	identity	log	fixed	fixed
normal	NO()	identity	log	-	-
normal family	NOF()	identity	log	identity	-
power exponential	PE()	identity	log	log	-
reverse Gumbel	RG()	identity	log	-	-
skew power exponential type 1	SEP1()	identity	log	identity	log
skew power exponential type 2	SEP2()	identity	log	identity	log
skew power exponential type 3	SEP3()	identity	log	log	log
skew power exponential type 4	SEP4()	identity	log	log	log
sinh-arcsinh	SHASH()	identity	log	log	log
skew t type 1	ST1()	identity	log	identity	log
skew t type 2	ST2()	identity	log	identity	log
skew t type 3	ST3()	identity	log	log	log
skew t type 4	ST4()	identity	log	log	log
skew t type 5	ST5()	identity	log	identity	log
t Family	TF()	identity	log	log	-
Weibull	WEI()	log	log	-	-
Weibull (PH)	WEI2()	log	log	-	-
Weibull (μ the mean)	WEI3()	log	log	-	-

Table 1.1: Continuous distributions implemented within the **gamlss** packages (with default link functions)

Distributions	R Name	μ	σ	ν
beta binomial	BB()	logit	log	-
binomial	BI()	logit	-	-
logarithmic	LG()	logit	-	-
Delaporte	DEL()	log	log	logit
negative binomial type I	NBI()	log	log	-
negative binomial type II	NBII()	log	log	-
Poisson	PO()	log	-	-
Poisson inverse Gaussian	PIG()	log	log	-
Sichel	SI()	log	log	identity
Sichel (μ the mean)	SICHEL()	log	log	identity
zero altered beta binomial	ZABB()	logit	log	logit
zero altered binomial	ZABI()	logit	logit	-
zero altered logarithmic	ZALG()	logit	logit	-
zero altered neg. binomial	ZANBI()	log	log	logit
zero altered poisson	ZAP()	log	logit	-
zero inflated beta binomial	ZIBB()	logit	log	logit
zero inflated binomial	ZIBI()	logit	logit	-
zero inflated neg. binomial	ZINBI()	log	log	logit
zero inflated poisson	ZIP()	log	logit	-
zero inflated poisson (μ the mean)	ZIP2()	log	logit	-
zero inflated poisson inv. Gaussian	ZIPIG()	log	log	logit

Table 1.2: Discrete distributions implemented within the **gamlss** packages (with default link functions)

beta inflated (at 0)	BE0I()	logit	log	logit	-
beta inflated (at 0)	BEINF0()	logit	logit	log	-
beta inflated (at 1)	BEZI()	logit	log	logit	-
beta inflated (at 1)	BEINF1()	logit	logit	log	-
beta inflated (at 0 and 1)	BEINF()	logit	logit	log	log
zero adjusted GA	ZAGA()	log	log	logit	-
zero adjusted IG	ZAIG()	log	log	logit	-

Table 1.3: Mixed distributions implemented within the **gamlss** packages (with default link functions)

by $\mathbf{NO}(\mu, \sigma)$, is

$$f_Y(y|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(y - \mu)^2}{2\sigma^2} \right] \quad (1.1)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$ and $\sigma > 0$. The mean of Y is given by $E(Y) = \mu$ and the variance of Y by $Var(Y) = \sigma^2$, so μ is the mean and σ is the standard deviation of Y .

Second parameterization (NO2)

$\mathbf{NO2}(\mu, \sigma)$ is a parameterization of the normal distribution where μ represents the mean and σ represents the variance of Y , i.e. $f_Y(y|\mu, \sigma) = (1/\sqrt{2\pi\sigma}) \exp[-(y - \mu)^2/(2\sigma)]$.

Normal family (of variance-mean relationships) (NOF)

The function $\mathbf{NOF}(\mu, \sigma, \nu)$ defines a normal distribution family with three parameters. The third parameter ν allows the variance of the distribution to be proportional to a power of the mean. The mean of $\mathbf{NOF}(\mu, \sigma, \nu)$ is equal to μ while the variance is equal to $Var(Y) = \sigma^2|\mu|^\nu$, so the standard deviation is $\sigma|\mu|^{\nu/2}$. The parametrization of the normal distribution given in the function $\mathbf{NOF}(\mu, \sigma, \nu)$ is

$$f(y|\mu, \sigma, \nu) = \frac{1}{\sqrt{2\pi\sigma}|\mu|^{\nu/2}} \exp \left[-\frac{(y - \mu)^2}{2\sigma^2|\mu|^\nu} \right] \quad (1.2)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $-\infty < \nu < \infty$.

The function $\mathbf{NOF}(\mu, \sigma, \nu)$ is appropriate for normally distributed regression type models where the variance of the response variable is proportional to a power of the mean. Models of this type are related to the "pseudo likelihood" models of Carroll and Rubert (1987) but here a proper likelihood is maximized. The ν parameter here is not designed to be modelled against explanatory variables but is a constant used as a device allowing us to model the variance mean relationship. Note that, due to the high correlation between the σ and ν parameters, the `mixed()` method argument is essential in the `gamlss()` fitting function. Alternatively ν can be estimated from its profile function, obtained using `gamlss` package function `prof.dev()`.

1.1.2 Logistic distribution (LO)

The logistic distribution is appropriate for moderately kurtotic data. The parameterization of the logistic distribution, denoted here as $\mathbf{LO}(\mu, \sigma)$, is given by

$$f_Y(y|\mu, \sigma) = \frac{1}{\sigma} \left\{ \exp \left[-\left(\frac{y - \mu}{\sigma} \right) \right] \right\} \left\{ 1 + \exp \left[-\left(\frac{y - \mu}{\sigma} \right) \right] \right\}^{-2} \quad (1.3)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$ and $\sigma > 0$, with $E(Y) = \mu$ and $Var(Y) = \pi^2\sigma^2/3$, Johnson *et al.* (1995) p 116.

1.1.3 Gumbel distribution (GU)

The Gumbel distribution is appropriate for moderately negative skew data. The pdf of the Gumbel distribution (or extreme value or Gompertz), denoted by $\mathbf{GU}(\mu, \sigma)$, is defined by

$$f_Y(y|\mu, \sigma) = \frac{1}{\sigma} \exp \left[\left(\frac{y - \mu}{\sigma} \right) - \exp \left(\frac{y - \mu}{\sigma} \right) \right] \quad (1.4)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$ and $\sigma > 0$, with $E(Y) = \mu - \gamma\sigma \simeq \mu - 0.57722\sigma$ and $Var(Y) = \pi^2\sigma^2/6 \simeq 1.64493\sigma^2$. See Crowder *et al.* (1991) p 17.

1.1.4 Reverse Gumbel distribution (RG)

The reverse Gumbel distribution, which is also called is the **type I extreme value distribution** is a special case of the generalized extreme value distribution, [see Johnson *et al.* (1995) p 2 and p 75]. The reverse Gumbel distribution is appropriate for moderately positive skew data. The pdf of the reverse Gumbel distribution, denoted by **RG**(μ, σ) is defined by

$$f_Y(y|\mu, \sigma) = \frac{1}{\sigma} \exp \left\{ - \left(\frac{y - \mu}{\sigma} \right) - \exp \left[- \frac{(y - \mu)}{\sigma} \right] \right\} \quad (1.5)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$ and $\sigma > 0$, with $E(Y) = \mu + \gamma\sigma \simeq \mu + 0.57722\sigma$ and $Var(Y) = \pi^2\sigma^2/6 \simeq 1.64493\sigma^2$. [Note that if $Y \sim RG(\mu, \sigma)$ and $W = -Y$, then $W \sim GU(-\mu, \sigma)$.]

1.2 Continuous three parameter distributions on \Re

1.2.1 Exponential Gaussian distribution (exGAUS)

The pdf of the ex-Gaussian distribution, denoted by **exGAUS**(μ, σ), is defined as

$$f_Y(y|\mu, \sigma, \nu) = \frac{1}{\nu} \exp \left[\frac{\mu - y}{\nu} + \frac{\sigma^2}{2\nu^2} \right] \Phi \left(\frac{y - \mu}{\sigma} - \frac{\sigma}{\nu} \right) \quad (1.6)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $\nu > 0$, and where Φ is the cdf of the standard normal distribution. Since $Y = Y_1 + Y_2$ where $Y_1 \sim N(\mu, \sigma^2)$ and $Y_2 \sim EX(\nu)$ are independent, the mean of Y is given by $E(Y) = \mu + \nu$ and the variance is given by $Var(Y) = \sigma^2 + \nu^2$. This distribution has also been called the lagged normal distribution, Johnson *et al.*, (1994), p172.

1.2.2 Power Exponential distribution (PE, PE2)

First parameterization (PE)

The power exponential distribution is suitable for leptokurtic as well as platykurtic data. The pdf of the power exponential family distribution, denoted by **PE**(μ, σ, ν), is defined by

$$f_Y(y|\mu, \sigma, \nu) = \frac{\nu \exp[-|\frac{z}{c}|^\nu]}{2c\sigma\Gamma(\frac{1}{\nu})} \quad (1.7)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $\nu > 0$ and where $z = (y - \mu)/\sigma$ and $c^2 = \Gamma(1/\nu)[\Gamma(3/\nu)]^{-1}$.

In this parameterization, used by Nelson (1991), $E(Y) = \mu$ and $Var(Y) = \sigma^2$. Note that $\nu = 1$ and $\nu = 2$ correspond to the Laplace (i.e. two sided exponential) and normal distributions respectively, while the uniform distribution is the limiting distribution as $\nu \rightarrow \infty$.

The cdf of Y is given by $F_Y(y) = \frac{1}{2} [1 + F_S(s)\text{sign}(z)]$ where $S = |z/c|^\nu$ has a gamma distribution with pdf $f_S(s) = s^{1/\nu} \exp(-s)/\Gamma(\frac{1}{\nu})$.

Second parameterization (PE2)

An alternative parameterization, the power exponential type 2 distribution, denoted by **PE2**(μ, σ, ν), is defined by

$$f_Y(y|\mu, \sigma, \nu) = \frac{\nu \exp[-|z|^\nu]}{2\sigma\Gamma(\frac{1}{\nu})} \quad (1.8)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $\nu > 0$ and where $z = (y - \mu)/\sigma$. Here $E(Y) = \mu$ and $Var(Y) = \sigma^2/c^2$, where $c^2 = \Gamma(1/\nu)[\Gamma(3/\nu)]^{-1}$.

See also Johnson *et al.*, 1995, volume 2, p195, equation (24.83) for a re-parameterized version by Subbotin (1923).

1.2.3 t family distribution (TF)

The t family distribution is suitable for modelling leptokurtic data, that is, data with higher kurtosis than the normal distribution. The pdf of the t family distribution, denoted here as **TF**(μ, σ, ν), is defined by

$$f_Y(y|\mu, \sigma, \nu) = \frac{1}{\sigma B(\frac{1}{2}, \frac{\nu}{2}) \nu^{\frac{1}{2}}} \left[1 + \frac{(y - \mu)^2}{\sigma^2 \nu} \right]^{-\frac{\nu+1}{2}} \quad (1.9)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $\nu > 0$, where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. The mean and variance of Y are given by $E(Y) = \mu$ and $Var(Y) = \sigma^2\nu/(\nu-2)$ when $\nu > 2$. Note that $T = (Y - \mu)/\sigma$ has a standard t distribution with ν degrees of freedom, given by Johnson *et al.* (1995), p 363, equation (28.2).

1.3 Continuous four parameter distributions on \Re

1.3.1 Exponential Generalized Beta type 2 distribution (EGB2)

The pdf of the exponential generalized beta type 2 distribution, denoted by **EGB2**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = e^{\nu z} \{ |\sigma| B(\nu, \tau) [1 + e^z]^{\nu+\tau} \}^{-1} \quad (1.10)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $-\infty < \sigma < \infty$, $\nu > 0$ and $\tau > 0$, and where $z = (y - \mu)/\sigma$, McDonald and Xu (1995), equation (3.3). Here $E(Y) = \mu + \sigma [\Psi(\nu) - \Psi(\tau)]$ and $Var(Y) = \sigma^2 [\Psi^{(1)}(\nu) + \Psi^{(1)}(\tau)]$, from McDonald (1996), p437.

1.3.2 Generalized t distribution (GT)

This pdf of the generalized t distribution, denoted by **GT**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \tau \left\{ 2\sigma\nu^{1/\tau} B(1/\tau, \nu) [1 + |z|^\tau/\nu]^{\nu+(1/\tau)} \right\}^{-1} \quad (1.11)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $\nu > 0$ and $\tau > 0$, and where $z = (y - \mu)/\sigma$, McDonald (1991) and McDonald and Newey (1988). Here $E(Y) = \mu$ and $Var(Y) = \sigma^2\nu^{2/\tau} B(\frac{3}{\tau}, \nu - \frac{2}{\tau})/B(\frac{1}{\tau}, \nu)$, from McDonald (1991) p274.

1.3.3 Johnson SU distribution (JSUo, JSU)

First parameterization (JSUo)

This is the original parameterization of the Johnson S_u distribution, Johnson (1949). The parameter ν determines the skewness of the distribution with $\nu > 0$ indicating negative skewness and $\nu < 0$ positive skewness. The parameter τ determines the kurtosis of the distribution. τ should be positive and most likely in the region above 1. As $\tau \rightarrow \infty$ the distribution approaches the normal density function. The distribution is appropriate for leptokurtotic data.

The pdf of the original Johnson's S_u , denoted here as **JSUo**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{\tau}{\sigma} \frac{1}{(r^2 + 1)^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} z^2 \right] \quad (1.12)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \nu < \infty$ and $\tau > 0$, and where

$$z = \nu + \tau \sinh^{-1}(r) = \nu + \tau \log \left[r + (r^2 + 1)^{\frac{1}{2}} \right], \quad (1.13)$$

where $r = (y - \mu)/\sigma$. Note that $Z \sim \mathbf{NO}(0, 1)$. Here $E(Y) = \mu - \sigma \omega^{1/2} \sinh(\nu/\tau)$ and $Var(Y) = \sigma^2 \frac{1}{2}(\omega - 1) [\omega \cosh(2\nu/\tau) + 1]$, where $\omega = \exp(1/\tau^2)$.

Second parameterization (JSU)

This is a reparameterization of the original Johnson S_u distribution, Johnson (1949), so that parameters μ and σ are the mean and the standard deviation of the distribution. The parameter ν determines the skewness of the distribution with $\nu > 0$ indicating positive skewness and $\nu < 0$ negative. The parameter τ determines the kurtosis of the distribution. τ should be positive and most likely in the region above 1. As $\tau \rightarrow \infty$ the distribution approaches the normal density function. The distribution is appropriate for leptokurtic data.

The pdf of the Johnson's S_u , denoted here as **JSU**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{\tau}{c\sigma} \frac{1}{(r^2 + 1)^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} z^2 \right] \quad (1.14)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \nu < \infty$, $\tau > 0$, and where

$$z = -\nu + \tau \sinh^{-1}(r) = -\nu + \tau \log \left[r + (r^2 + 1)^{\frac{1}{2}} \right], \quad (1.15)$$

$$r = \frac{y - (\mu + c\sigma w^{\frac{1}{2}} \sinh \Omega)}{c\sigma},$$

$$c = \left\{ \frac{1}{2}(w - 1) [w \cosh(2\Omega) + 1] \right\}^{-\frac{1}{2}},$$

$w = \exp(1/\tau^2)$ and $\Omega = -\nu/\tau$. Note that $Z \sim \mathbf{NO}(0, 1)$. Here $E(Y) = \mu$ and $Var(Y) = \sigma^2$.

1.3.4 Normal-Exponential- t distribution (NET)

The NET distribution is a four parameter continuous distribution, although in **gamlss** it is used as a two parameter distribution with the other two of its parameters fixed. It was introduced by Rigby and Stasinopoulos (1994) as a robust method of fitting the mean and the scale parameters of a symmetric distribution as functions of explanatory variables. The NET distribution is the abbreviation of the Normal-Exponential-Student- t distribution and is denoted by **NET**(μ, σ, ν, τ), for given values for ν and τ . It is normal up to ν , exponential from ν to τ and Student- t with $(\nu\tau - 1)$ degrees of freedom after τ . Fitted parameters are the first two parameters, μ and σ . Parameters ν and τ may be chosen and fixed by the user. Alternatively estimates of the third and forth parameters can be obtained, using the **gamlss** function **prof.dev()**.

The pdf of the normal exponential t distribution, denoted here as **NET**(μ, σ, ν, τ), is given by Rigby and Stasinopoulos (1994) and defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{c}{\sigma} \begin{cases} \exp\left\{-\frac{z^2}{2}\right\}, & \text{when } |z| \leq \nu \\ \exp\left\{-\nu|z| + \frac{\nu^2}{2}\right\}, & \text{when } \nu < |z| \leq \tau \\ \exp\left\{-\nu\tau \log\left(\frac{|z|}{\tau}\right) - \nu\tau + \frac{\nu^2}{2}\right\}, & \text{when } |z| > \tau \end{cases} \quad (1.16)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $\nu > 1$, $\tau > \nu$ ¹, and where $z = (y - \mu)/\sigma$ and $c = (c_1 + c_2 + c_3)^{-1}$, where $c_1 = \sqrt{2\pi} [1 - 2\Phi(-\nu)]$, $c_2 = \frac{2}{\nu} \exp\left\{-\frac{\nu^2}{2}\right\}$ and $c_3 = \frac{2}{(\nu\tau-1)\nu} \exp\left\{-\nu\tau + \frac{\nu^2}{2}\right\}$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Here μ is the mean of Y .

1.3.5 Sinh-Arcsinh (SHASH)

The pdf of the Sinh-Arcsinh distribution, denoted by **SHASH**(μ, σ, ν, τ), Jones(2005), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{c}{\sqrt{2\pi}\sigma(1+r^2)^{1/2}} e^{-z^2/2} \quad (1.17)$$

where

$$z = \frac{1}{2} \left\{ \exp[\tau \sinh^{-1}(r)] - \exp[-\nu \sinh^{-1}(r)] \right\}$$

and

$$c = \frac{1}{2} \left\{ \tau \exp[\tau \sinh^{-1}(r)] + \nu \exp[-\nu \sinh^{-1}(r)] \right\}$$

and $r = (y - \mu)/\sigma$ for $-\infty < y < \infty$, where $-\infty < \mu < +\infty$, $\sigma > 0$, $\nu > 0$ and $\tau > 0$. Note $\sinh^{-1}(r) = \log(u)$ where $u = r + (r^2 + 1)^{1/2}$. Hence $z = \frac{1}{2}(u^\tau - u^{-\nu})$. Note that $Z \sim \mathbf{NO}(0, 1)$. Hence μ is the median of Y .

¹since NET involves the Student- t distribution with $(\nu\tau-1)$ degrees of freedom

1.3.6 Skew Exponential Power type 1 distribution (SEP1)

The pdf of the skew exponential power type 1 distribution, denoted by **SEP1**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{2}{\sigma} f_{Z_1}(z) F_{Z_1}(\nu z) \quad (1.18)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \nu < \infty$ and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and f_{Z_1} and F_{Z_1} are the pdf and cdf of $Z_1 \sim PE2(0, \tau^{1/\tau}, \tau)$, a power exponential type 2 distribution with $f_{Z_1}(z) = \alpha^{-1} \exp[-|z|^\tau/\tau]$, where $\alpha = 2\tau^{(1/\tau)-1}\Gamma(1/\tau)$. This distribution was introduced by Azzalini (1986) as his type I distribution.

Here $E(Y) = \mu + \sigma E(Z)$ and $Var(Y) = \sigma^2 V(Z) = \sigma^2 \{E(Z^2) - [E(Z)]^2\}$ where $Z = (Y - \mu)/\sigma$ and $E(Z) = \text{sign}(\nu)\tau^{1/\tau} [\Gamma(\frac{2}{\tau})/\Gamma(\frac{1}{\tau})] pBEo\left(\frac{\nu^\tau}{1+\nu^\tau}, \frac{1}{\tau}, \frac{2}{\tau}\right)$, and $E(Z^2) = \tau^{2/\tau}\Gamma(\frac{3}{\tau})/\Gamma(\frac{1}{\tau})$, where $pBEo(q, a, b)$ is the cdf of an original beta distribution $BEo(a, b)$ evaluated at q , Azzalini (1986), p202-203.

The skew normal type 1 distribution, denoted by **SN1**(μ, σ, ν), a special case of **SEP1**(μ, σ, ν, τ) given by $\tau = 2$, has mean and variance given by $E(Y) = \mu + \sigma \text{sign}(\nu) \{2\nu^2 / [\pi(1 + \nu^2)]\}^{1/2}$ and $Var(Y) = \sigma^2 \{1 - 2\nu^2 / [\pi(1 + \nu^2)]\}$, Azzalini (1985), p174. Note that **SN1** is not currently implemented as a specific distribution, but can be obtained by fixing $\tau = 2$ in **SEP1** using the arguments `tau.start=2, tau.fix=TRUE` in `gamlss()`.

1.3.7 Skew Exponential Power type 2 distribution (SEP2)

The pdf of the skew exponential power type 2 distribution, denoted by **SEP2**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{2}{\sigma} f_{Z_1}(z) \Phi(\omega) \quad (1.19)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \nu < \infty$, and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and $\omega = \text{sign}(z)|z|^{\tau/2}\nu\sqrt{2/\tau}$ and f_{Z_1} is the pdf of $Z_1 \sim PE2(0, \tau^{1/\tau}, \tau)$ and $\Phi(\omega)$ is the cdf of a standard normal variable evaluated at ω .

This distribution was introduced by Azzalini (1986) as his type II distribution and was further developed by DiCiccio and Monti (2004). The parameter ν determines the skewness of the distribution with $\nu > 0$ indicating positive skewness and $\nu < 0$ negative. The parameter τ determines the kurtosis of the distribution, with $\tau > 2$ for platykurtic data and $\tau < 2$ for leptokurtic.

Here $E(Y) = \mu + \sigma E(Z)$ and $Var(Y) = \sigma^2 V(Z)$ where

$$E(Z) = \frac{2\tau^{1/\tau}\nu}{\sqrt{\pi}\Gamma(\frac{1}{\tau})(1+\nu^2)^{(2/\tau)+(1/2)}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2}{\tau} + n + \frac{1}{2})}{(2n+1)!!} \left(\frac{2\nu^2}{1+\nu^2}\right)^n \quad (1.20)$$

and $E(Z^2) = \tau^{2/\tau}\Gamma(\frac{3}{\tau})/\Gamma(\frac{1}{\tau})$, where $(2n+1)!! = 1.3.5...(2n-1)$, DiCiccio and Monti (2004), p439.

For $\tau = 2$ the **SEP2**(μ, σ, ν, τ) distribution is the skew normal type 1 distribution, Azzalini (1985), denoted by **SN1**(μ, σ, ν), while for $\nu = 1$ and $\tau = 2$ the **SEP2**(μ, σ, ν, τ) distribution is the normal density function, **NO**(μ, σ).

1.3.8 Skew Exponential Power type 3 distribution (SEP3)

This is a "spliced-scale" distribution with pdf, denoted by **SEP3**(μ, σ, ν, τ), defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{c}{\sigma} \left\{ \exp \left[-\frac{1}{2} |\nu z|^\tau \right] I(y < \mu) + \exp \left[-\frac{1}{2} \left| \frac{z}{\nu} \right|^\tau \right] I(y \geq \mu) \right\} \quad (1.21)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $\nu > 0$, and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and $c = \nu\tau / [(1 + \nu^2)2^{1/\tau}\Gamma(\frac{1}{\tau})]$, Fernandez, Osiewalski and Steel (1995). Note that $I()$ is an indicator function, where $I(u) = 1$ if u is true and $I(u) = 0$ if u is false.

Note that μ is the mode of Y . Here $E(Y) = \mu + \sigma E(Z)$ and $Var(Y) = \sigma^2 V(Z)$ where $E(Z) = 2^{1/\tau}\Gamma(\frac{2}{\tau})(\nu - \frac{1}{\nu})/\Gamma(\frac{1}{\tau})$ and $E(Z^2) = 2^{2/\tau}\Gamma(\frac{3}{\tau})(\nu^3 + \frac{1}{\nu^3})/[\Gamma(\frac{1}{\tau})(\nu + \frac{1}{\nu})]$, Fernandez, Osiewalski and Steel (1995), p1333, eqns. (12) and (13).

The skew normal type 2 distribution, Johnson *et al.* (1994) p173, denoted by **SN2**(μ, σ, ν), (or two-piece normal) is a special case of **SEP3**(μ, σ, ν, τ) given by $\tau = 2$.

1.3.9 Skew Exponential Power type 4 distribution (SEP4)

This is a "spliced-shape" distribution with pdf, denoted by **SEP4**(μ, σ, ν, τ), defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{c}{\sigma} \{ \exp[-|z|^\nu] I(y < \mu) + \exp[-|z|^\tau] I(y \geq \mu) \} \quad (1.22)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $\nu > 0$, and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and $c = [\Gamma(1 + \frac{1}{\nu}) + \Gamma(1 + \frac{1}{\tau})]^{-1}$, Jones (2005). Note that μ is the mode of Y .

Here $E(Y) = \mu + \sigma E(Z)$ and $Var(Y) = \sigma^2 V(Z)$ where $E(Z) = c [\frac{1}{\tau}\Gamma(\frac{2}{\tau}) - \frac{1}{\nu}\Gamma(\frac{2}{\nu})]$ and $E(Z^2) = c [\frac{1}{\nu}\Gamma(\frac{3}{\nu}) + \frac{1}{\tau}\Gamma(\frac{3}{\tau})]$.

1.3.10 Skew t type 1 distribution (ST1)

The pdf of the skew t type 1 distribution, denoted by **ST1**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{2}{\sigma} f_{Z_1}(z) F_{Z_1}(\nu z) \quad (1.23)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \nu < \infty$ and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and f_{Z_1} and F_{Z_1} are the pdf and cdf of $Z \sim TF(0, 1, \tau)$, a t distribution with $\tau > 0$ degrees of freedom, with τ treated as a continuous parameter. This distribution is in the form of a type I distribution of Azzalini (1986).

1.3.11 Skew t type 2 distribution (ST2)

The pdf of the skew t type 2 distribution, denoted by **ST2**(μ, σ, ν, τ), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{2}{\sigma} f_{Z_1}(z) F_{Z_2}(w) \quad (1.24)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \nu < \infty$, and $\tau > 0$, and where $z = (y - \mu)/\sigma$, $w = \nu\lambda^{1/2}z$ and $\lambda = (\tau + 1)/(\tau + z^2)$ and f_{Z_1} is the pdf of $Z_1 \sim TF(0, 1, \tau)$ and F_{Z_1} is the cdf of $Z_2 \sim TF(0, 1, \tau + 1)$. This distribution is the univariate case of the multivariate skew t distribution introduced by Azzalini and Capitanio (2003).

Here the mean and variance of Y are given by $E(Y) = \mu + \sigma E(Z)$ and $Var(Y) = \sigma^2 V(Z)$ where $E(Z) = \nu\tau^{1/2}\Gamma(\frac{\tau-1}{2})/[\pi^{1/2}(1 + \nu^2)^{1/2}\Gamma(\frac{\tau}{2})]$ for $\tau > 1$ and $E(Z^2) = \tau/(\tau - 2)$ for $\tau > 2$, Azzalini and Capitanio (2003), p382.

1.3.12 Skew t type 3 distribution (ST3)

This is a "spliced-scale" distribution with pdf, denoted by $ST3(\mu, \sigma, \nu, \tau)$, defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{c}{\sigma} \left\{ 1 + \frac{z^2}{\tau} \left[\nu^2 I(y < \mu) + \frac{1}{\nu^2} I(y \geq \mu) \right] \right\} \quad (1.25)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $\nu > 0$, and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and $c = 2\nu / [\sigma (1 + \nu^2) B(\frac{1}{2}, \frac{\tau}{2}) \tau^{1/2}]$, Fernandez and Steel (1998).

Note that μ is the mode of Y . The mean and variance of Y are given by $E(Y) = \mu + \sigma E(Z)$ and $Var(Y) = \sigma^2 V(Z)$ where $E(Z) = 2\tau^{1/2}(\nu^2 - 1) / [(\tau - 1)B(\frac{1}{2}, \frac{\tau}{2}) \nu]$ and $E(Z^2) = \tau(\nu^3 + \frac{1}{\nu^3}) / [(\tau - 2)(\nu + \frac{1}{\nu})]$, Fernandez and Steel (1998), p360, eqn. (5).

1.3.13 Skew t type 4 distribution (ST4)

This is a "spliced-shape" distribution with pdf, denoted by $ST4(\mu, \sigma, \nu, \tau)$, defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{c}{\sigma} \left\{ \left[1 + \frac{z^2}{\nu} \right]^{-(\nu+1)/2} I(y < \mu) + \left[1 + \frac{z^2}{\tau} \right]^{-(\tau+1)/2} I(y \geq \mu) \right\} \quad (1.26)$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $\nu > 0$ and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and $c = 2[\nu^{1/2} B(\frac{1}{2}, \frac{\nu}{2}) + \tau^{1/2} B(\frac{1}{2}, \frac{\tau}{2})]^{-1}$.

Here $E(Y) = \mu + \sigma E(Z)$ and $Var(Y) = \sigma^2 V(Z)$ where $E(Z) = c \left[\frac{1}{\tau-1} - \frac{1}{\nu-1} \right]$, provided $\nu > 1$ and $\tau > 1$, and $E(Z^2) = \frac{c}{2} \left\{ \left[\tau^{3/2} B(\frac{1}{2}, \frac{\tau}{2}) / (\tau - 2) \right] + \left[\nu^{3/2} B(\frac{1}{2}, \frac{\nu}{2}) / (\nu - 2) \right] \right\}$, provided $\nu > 2$ and $\tau > 2$.

1.3.14 Skew t type 5 distribution (ST5)

The pdf of the skew t distribution type 5, denoted by $\mathbf{ST5}(\mu, \sigma, \nu, \tau)$, Jones and Faddy (2003), is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{c}{\sigma} \left[1 + \frac{z}{(a+b+z^2)^{1/2}} \right]^{a+1/2} \left[1 - \frac{z}{(a+b+z^2)^{1/2}} \right]^{b+1/2}$$

for $-\infty < y < \infty$, where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \nu < \infty$ and $\tau > 0$, and where $z = (y - \mu)/\sigma$ and $c = [2^{a+b-1}(a+b)^{1/2} B(a, b)]^{-1}$ and $\nu = (a-b)/[ab(a+b)]^{1/2}$ and $\tau = 2/(a+b)$.

Here $E(Y) = \mu + \sigma E(Z)$ where $E(Z) = (a-b)(a+b)^{1/2} \Gamma(a - \frac{1}{2}) \Gamma(a - \frac{1}{2}) / [2\Gamma(a)\Gamma(b)]$ and $Var(Y) = \sigma^2 V(Z)$ where $E(Z^2) = (a+b) [(a-b)^2 + a+b-2] / [4(a-1)(b-1)]$, Jones and Faddy (2003), p162.

1.4 Continuous one parameter distribution in \mathbb{R}^+

1.4.1 Exponential distribution (EXP)

This is the only one parameter continuous distribution in **gamlss** packages. The exponential distribution is appropriate for moderately positive skew data. The parameterization of the exponential distribution, denoted here as $\mathbf{EXP}(\mu)$, is defined by

$$f_Y(y|\mu) = \frac{1}{\mu} \exp \left\{ -\frac{y}{\mu} \right\} \quad (1.27)$$

for $y > 0$, where $\mu > 0$ and where $E(Y) = \mu$ and $Var(Y) = \mu^2$.

1.5 Continuous two parameter distribution in \mathbb{R}^+

1.5.1 Gamma distribution (GA)

The gamma distribution is appropriate for positively skew data. The pdf of the gamma distribution, denoted by **GA**(μ, σ), is defined by

$$f_Y(y|\mu, \sigma) = \frac{1}{(\sigma^2\mu)^{1/\sigma^2}} \frac{y^{\frac{1}{\sigma^2}-1} e^{-y/(\sigma^2\mu)}}{\Gamma(1/\sigma^2)} \quad (1.28)$$

for $y > 0$, where $\mu > 0$ and $\sigma > 0$. Here $E(Y) = \mu$ and $Var(Y) = \sigma^2\mu^2$. This a reparameterization of Johnson *et al.* (1994) p 343 equation (17.23) obtained by setting $\sigma^2 = 1/\alpha$ and $\mu = \alpha\beta$.

1.5.2 Log Normal distribution (LOGNO, LNO)

Log Normal distribution (LOGNO)

The log-normal distribution is appropriate for positively skew data. The pdf of the log-normal distribution, denoted by **LOGNO**(μ, σ), is defined by

$$f_Y(y|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{y} \exp \left\{ -\frac{[\log(y) - \mu]^2}{2\sigma^2} \right\} \quad (1.29)$$

for $y > 0$, where $\mu > 0$ and $\sigma > 0$. Here $E(Y) = \omega^{1/2}e^\mu$ and $Var(Y) = \omega(\omega - 1)e^{2\mu}$, where $\omega = \exp(\sigma^2)$.

Log normal family (i.e. original Box-Cox) (LNO)

The **gamlss** function **LNO**(μ, σ, ν) allows the use of the Box-Cox power transformation approach, Box and Cox (1964), where the transformation $Y(\nu)$ is applied to Y in order to remove skewness, where $Z = (Y^\nu - 1)/\nu$ (if $\nu \neq 0$) + $\log(Y)$ (if $\nu = 0$). The transformed variable Z is then assumed to have a normal $NO(\mu, \sigma)$ distribution. The resulting distribution for Y is denoted by **LNO**(μ, σ, ν). When $\nu = 0$, this results in the distribution in equation (1.29). For values of ν different from zero we have the resulting three parameter distribution

$$f_Y(y|\mu, \sigma, \nu) = \frac{y^{\nu-1}}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(z - \mu)^2}{2\sigma^2} \right] \quad (1.30)$$

for $y > 0$, where $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$, and where $z = (y^\nu - 1)/\nu$ (if $\nu \neq 0$) + $\log(y)$ (if $\nu = 0$). The distribution in (1.30) can be fitted for fixed ν only, e.g. $\nu = 0.5$, using the following arguments of **gamlss**(): **family=LNO**, **nu.fix=TRUE**, **nu.start=0.5**. If ν is unknown, it can be estimated from its profile likelihood. Alternatively instead of (1.30), the more orthogonal parameterization of (1.30) given by the BCCG distribution in Section 1.6.1 can be used.

1.5.3 Inverse Gaussian distribution (IG)

The inverse Gaussian distribution is appropriate for highly positive skew data. The pdf of the inverse Gaussian distribution, denoted by **IG**(μ, σ) is defined by

$$f_Y(y|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2 y^3}} \exp \left[-\frac{1}{2\mu^2\sigma^2 y} (y - \mu)^2 \right] \quad (1.31)$$

for $y > 0$, where $\mu > 0$ and $\sigma > 0$ with $E(Y) = \mu$ and $Var(Y) = \sigma^2 \mu^3$. This is a reparameterization of Johnson *et al.* (1994) p 261 equation (15.4a), obtained by setting $\sigma^2 = 1/\lambda$.

1.5.4 Weibull distribution (WEI, WEI2, WEI3)

First parameterization (WEI)

There are three version of the two parameter Weibull distribution implemented into the **gamlss** package. The first, denoted by **WEI**(μ, σ), has the following parameterization

$$f_Y(y|\mu, \sigma) = \frac{\sigma y^{\sigma-1}}{\mu^\sigma} \exp \left[- \left(\frac{y}{\mu} \right)^\sigma \right] \quad (1.32)$$

for $y > 0$, where $\mu > 0$ and $\sigma > 0$, [see Johnson *et al.* (1994) p629]. The mean and the variance of Y in this parameterization (1.32) of the two parameter Weibull are given by $E(Y) = \mu \Gamma(\frac{1}{\sigma} + 1)$ and $Var(Y) = \mu^2 \left\{ \Gamma(\frac{2}{\sigma} + 1) - [\Gamma(\frac{1}{\sigma} + 1)]^2 \right\}$, from Johnson *et al.* (1994) p632. Although the parameter μ is a scale parameter, it also affects the mean of Y . The median of Y is $m_Y = \mu(\log 2)^{1/\sigma}$, see Johnson *et al.* (1994), p630.

Second parameterization (WEI2)

The second parameterization of the Weibull distribution, denoted by **WEI2**(μ, σ), is defined as

$$f_Y(y|\mu, \sigma) = \sigma \mu y^{\sigma-1} e^{-\mu y^\sigma} \quad (1.33)$$

for $y > 0$, where $\mu > 0$ and $\sigma > 0$, Johnson *et al.* (1994), p686. The mean of Y in this parameterization (1.33) is $E(Y) = \mu^{-1/\sigma} \Gamma(\frac{1}{\sigma} + 1)$ and the variance of Y is $Var(Y) = \mu^{-2/\sigma} \left\{ \Gamma(\frac{2}{\sigma} + 1) - [\Gamma(\frac{1}{\sigma} + 1)]^2 \right\}$.

In the second parameterization of the Weibull distribution the two parameters μ and σ are highly correlated, so the RS method of fitting is very slow and therefore the **CG**() method of fitting should be used.

Third parameterization (WEI3)

This is a parameterization of the Weibull distribution where μ is the mean of the distribution. This parameterization of the Weibull distribution, denoted by **WEI3**(μ, σ), is defined as

$$f_Y(y|\mu, \sigma) = \frac{\sigma}{\beta} \left(\frac{y}{\beta} \right)^{\sigma-1} \exp \left\{ - \left(\frac{y}{\beta} \right)^\sigma \right\} \quad (1.34)$$

for $y > 0$, where $\mu > 0$ and $\sigma > 0$ and where $\beta = \mu/\Gamma(\frac{1}{\sigma} + 1)$. The mean of Y is given by $E(Y) = \mu$ and the variance $Var(Y) = \mu^2 \left\{ \Gamma(\frac{2}{\sigma} + 1) [\Gamma(\frac{1}{\sigma} + 1)]^{-2} - 1 \right\}$.

1.6 Continuous three parameter distribution in \mathbb{R}^+

1.6.1 Box-Cox Cole and Green distribution (BCCG)

The Box-Cox Cole and Green distribution is suitable for positively or negatively skew data. Let $Y > 0$ be a positive random variable having a Box-Cox Cole and Green distribution, denoted

here as **BCCG**(μ, σ, ν), defined through the transformed random variable Z given by

$$Z = \begin{cases} \frac{1}{\sigma\nu} \left[\left(\frac{Y}{\mu} \right)^\nu - 1 \right], & \text{if } \nu \neq 0 \\ \frac{1}{\sigma} \log\left(\frac{Y}{\mu}\right), & \text{if } \nu = 0 \end{cases} \quad (1.35)$$

for $0 < Y < \infty$, where $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$, and where the random variable Z is assumed to follow a truncated standard normal distribution. The condition $0 < Y < \infty$ (required for Y^ν to be real for all ν) leads to the condition $-1/(\sigma\nu) < Z < \infty$ if $\nu > 0$ and $-\infty < Z < -1/(\sigma\nu)$ if $\nu < 0$, which necessitates the truncated standard normal distribution for Z .

Hence the pdf of Y is given by

$$f_Y(y) = \frac{y^{\nu-1} \exp(-\frac{1}{2}z^2)}{\mu^\nu \sigma \sqrt{2\pi} \Phi(\frac{1}{\sigma|\nu|})} \quad (1.36)$$

where z is given by (1.35) and $\Phi()$ is the cumulative distribution function (cdf) of a standard normal distribution.

If the truncation probability $\Phi(-\frac{1}{\sigma|\nu|})$ is negligible, the variable Y has median μ . The parameterization (1.35) was used by Cole and Green (1992) who assumed a standard normal distribution for Z and assumed that the truncation probability was negligible.

1.6.2 Generalized gamma distribution (GG, GG2)

First parameterization (GG)

The specific parameterization of the generalized gamma distribution used here and denoted by **GG**(μ, σ, ν) was used by Lopatzidis and Green (2000), and is defined as

$$f_Y(y|\mu, \sigma, \nu) = \frac{|\nu| \theta^\theta z^\theta \exp\{-\theta z\}}{\Gamma(\theta)y} \quad (1.37)$$

for $y > 0$, where $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$ and where $z = (y/\mu)^\nu$ and $\theta = 1/(\sigma^2\nu^2)$.

The mean and variance of Y are given by $E(Y) = \mu \Gamma(\theta + \frac{1}{\nu}) / [\theta^{1/\nu} \Gamma(\theta)]$ and $Var(Y) = \mu^2 \left\{ \Gamma(\theta) \Gamma(\theta + \frac{2}{\nu}) - [\Gamma(\theta + \frac{1}{\nu})]^2 \right\} / \left\{ \theta^{2/\nu} [\Gamma(\theta)]^2 \right\}$. Note that **GG2** is not currently implemented in **gamlss**.

Second parameterization (GG2)

A second parameterization, given by Johnson *et al.*, (1995), p401, denoted by **GG2**(μ, σ, ν), is defined as

$$f_Y(y|\mu, \sigma, \nu) = \frac{|\mu| y^{\mu\nu-1}}{\Gamma(\nu) \sigma^{\mu\nu}} \exp\left\{-\left(\frac{y}{\sigma}\right)^\mu\right\} \quad (1.38)$$

for $y > 0$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $\nu > 0$.

The mean and variance of $Y \sim \mathbf{GG2}(\mu, \sigma, \nu)$ can be obtained from those of **GG**(μ, σ, ν) since $\mathbf{GG}(\mu, \sigma, \nu) \equiv \mathbf{GG2}(\nu, \mu\theta^{-1/\nu}, \theta)$ and $\mathbf{GG2}(\mu, \sigma, \nu) \equiv \mathbf{GG}(\sigma\nu^{1/\mu}, [\mu^2\nu]^{-1/2}, \mu)$.

1.6.3 Generalized inverse Gaussian distribution (GIG)

The parameterization of the generalized inverse Gaussian distribution, denoted by $\mathbf{GIG}(\mu, \sigma, \nu)$, is defined as

$$f_Y(y|\mu, \sigma, \nu) = \left(\frac{c}{\mu}\right)^\nu \left[\frac{y^{\nu-1}}{2K_\nu\left(\frac{1}{\sigma^2}\right)}\right] \exp\left[-\frac{1}{2\sigma^2}\left(\frac{cy}{\mu} + \frac{\mu}{cy}\right)\right] \quad (1.39)$$

for $y > 0$, where $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$, where $c = [K_{\nu+1}(1/\sigma^2)] [K_\nu(1/\sigma^2)]^{-1}$ and $K_\lambda(t) = \frac{1}{2} \int_0^\infty x^{\lambda-1} \exp\{-\frac{1}{2}t(x + x^{-1})\} dx$.

Here $E(Y) = \mu$ and $Var(Y) = \mu^2 [2\sigma^2(\nu + 1)/c + 1/c^2 - 1]$. $\mathbf{GIG}(\mu, \sigma, \nu)$ is a reparameterization of the generalized inverse Gaussian distribution of Jorgensen (1982). Note also that $\mathbf{GIG}(\mu, \sigma, -0.5) \equiv \mathbf{IG}(\mu, \sigma\mu^{-1/2})$ a reparameterization of the inverse Gaussian distribution.

1.6.4 Zero adjusted Gamma distribution (ZAGA)

The zero adjusted Gamma distribution is appropriate when the response variable Y takes values from zero to infinity including zero, i.e. $[0, \infty)$. Hence $Y = 0$ has non zero probability ν . The pdf of the zero adjusted Gamma distribution, denoted by $\mathbf{ZAGA}(\mu, \sigma, \nu)$, is defined by

$$f_Y(y|\mu, \sigma, \nu) = \begin{cases} \nu & \text{if } y = 0 \\ (1 - \nu) \left[\frac{1}{(\sigma^2\mu)^{1/\sigma^2}} \frac{y^{\frac{1}{\sigma^2}-1} e^{-y/(\sigma^2\mu)}}{\Gamma(1/\sigma^2)} \right] & \text{if } y > 0 \end{cases} \quad (1.40)$$

for $0 \leq y < \infty$, where $0 < \nu < 1$, $\mu > 0$ and $\sigma > 0$ with $E(Y) = (1 - \nu)\mu$ and $Var(Y) = (1 - \nu)\mu^2(\nu + \sigma^2)$.

1.6.5 Zero adjusted Inverse Gaussian distribution (ZAIG)

The zero adjusted inverse Gaussian distribution is appropriate when the response variable Y takes values from zero to infinity including zero, i.e. $[0, \infty)$. Hence $Y = 0$ has non zero probability ν . The pdf of the zero adjusted inverse Gaussian distribution, denoted by $\mathbf{ZAIG}(\mu, \sigma, \nu)$, is defined by

$$f_Y(y|\mu, \sigma, \nu) = \begin{cases} \nu & \text{if } y = 0 \\ (1 - \nu) \frac{1}{\sqrt{2\pi\sigma^2 y^3}} \exp\left[-\frac{1}{2\mu^2\sigma^2 y} (y - \mu)^2\right] & \text{if } y > 0 \end{cases} \quad (1.41)$$

for $0 \leq y < \infty$, where $0 < \nu < 1$, $\mu > 0$ and $\sigma > 0$ with $E(Y) = (1 - \nu)\mu$ and $Var(Y) = (1 - \nu)\mu^2(\nu + \mu\sigma^2)$.

1.7 Continuous four parameter distribution in \mathcal{R}^+

1.7.1 Box-Cox t distribution (BCT)

Let Y be a positive random variable having a Box-Cox t distribution, Rigby and Stasinopoulos (2006), denoted by $\mathbf{BCT}(\mu, \sigma, \nu, \tau)$, defined through the transformed random variable Z given by (1.35), where the random variable Z is assumed to follow a truncated t distribution with degrees of freedom, $\tau > 0$, treated as a continuous parameter.

The pdf of Y , a $\mathbf{BCT}(\mu, \sigma, \nu, \tau)$ random variable, is given by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{y^{\nu-1} f_T(z)}{\mu^\nu \sigma F_T(\frac{1}{\sigma|\nu|})} \quad (1.42)$$

for $y > 0$, where $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$, and where z is given by (1.35) and $f_T(t)$ and $F_T(t)$ are respectively the pdf and cumulative distribution function of a random variable T having a standard t distribution with degrees of freedom parameter $\tau > 0$, ie $T \sim t_\tau \equiv \mathbf{TF}(0,1,\tau)$. If the truncation probability $F_T(-\frac{1}{\sigma|\nu|})$ is negligible, the variable Y has median μ .

1.7.2 Box-Cox power exponential distribution (BCPE)

Let Y be a positive random variable having a Box-Cox power exponential distribution, Rigby and Stasinopoulos (2004), denoted by $\mathbf{BCPE}(\mu, \sigma, \nu, \tau)$, defined through the transformed random variable Z given by (1.35), where the random variable Z is assumed to follow a truncated standard power exponential distribution with power parameter, $\tau > 0$, treated as a continuous parameter.

The pdf of Y , a $\mathbf{BCPE}(\mu, \sigma, \nu, \tau)$ random variable, is given by (1.42), where $f_T(t)$ and $F_T(t)$ are respectively the pdf and cumulative distribution function of a variable T having a standard power exponential distribution, $T \sim \mathbf{PE}(0,1,\tau)$. If the truncation probability $F_T(-\frac{1}{\sigma|\nu|})$ is negligible, the variable Y has median μ .

1.7.3 Generalized Beta type 2 distribution (GB2)

This pdf of the generalized beta type 2 distribution, denoted by $GB2(\mu, \sigma, \nu, \tau)$, is defined by

$$\begin{aligned} f_Y(y|\mu, \sigma, \nu, \tau) &= |\sigma| y^{\sigma\nu-1} \left\{ \mu^{\sigma\nu} B(\nu, \tau) [1 + (y/\mu)^\sigma]^{\nu+\tau} \right\}^{-1} \\ &= \frac{\Gamma(\nu + \tau)}{\Gamma(\nu)\Gamma(\tau)} \frac{\sigma(y/\mu)^{\sigma\nu}}{y [1 + (y/\mu)^\sigma]^{\nu+\tau}} \end{aligned} \quad (1.43)$$

for $y > 0$, where $\mu > 0$, $-\infty < \sigma < \infty$, $\nu > 0$ and $\tau > 0$, McDonald and Xu (1995), equation (2.7). The mean and variance of Y are given by $E(Y) = \mu B(\nu + \frac{1}{\sigma}, \tau - \frac{1}{\sigma}) / B(\nu, \tau)$ for $-\nu < \frac{1}{\sigma} < \tau$ and $E(Y^2) = \mu^2 B(\nu + \frac{2}{\sigma}, \tau - \frac{2}{\sigma}) / B(\nu, \tau)$ for $-\nu < \frac{2}{\sigma} < \tau$, McDonald (1996), p434. Note the by setting $\nu = 1$ in 1.43 we obtain the *Burr* distribution:

$$f_Y(y|\mu, \sigma, \tau) = \frac{\tau \sigma (y/\mu)^\sigma}{y [1 + (y/\mu)^\sigma]^{\tau+1}}. \quad (1.44)$$

By setting $\sigma = 1$ in 1.43 we obtain the *Generalized Pareto* distribution:

$$f_Y(y|\mu, \nu, \tau) = \frac{\Gamma(\nu + \tau)}{\Gamma(\nu)\Gamma(\tau)} \frac{(\mu^\tau y^{\nu-1})}{(y + \mu)^{\nu+\tau}}. \quad (1.45)$$

1.8 Continuous two parameter distribution in $\mathbb{R}[0, 1]$

1.8.1 Beta distribution (BE, BEo)

The beta distribution is appropriate when the response variable takes values in a known restricted range, excluding the endpoints of the range. Appropriate standardization can be applied to make the range of the response variable $(0,1)$, i.e. from zero to one excluding the endpoints. Note that $0 < Y < 1$ so values $Y = 0$ and $Y = 1$ have zero density under the model.

First parameterization (BEo)

The original parameterization of the beta distribution, denoted by $BEo(\mu, \sigma)$, has pdf given by $f_Y(y|\mu, \sigma) = \frac{1}{B(\mu, \sigma)} y^{\mu-1}(1-y)^{\sigma-1}$ for $0 < y < 1$, with parameters $\mu > 0$ and $\sigma > 0$. Here $E(Y) = \mu/(\mu + \sigma)$ and $Var(Y) = \mu\sigma(\mu + \sigma)^{-2}(\mu + \sigma + 1)^{-1}$.

Second parameterization (BE)

In the second parameterization of the beta distribution below the parameters μ and σ are location and scale parameters that relate to the mean and standard deviation of Y . The pdf of the beta distribution, denoted by $\mathbf{BE}(\mu, \sigma)$, is defined by

$$f_Y(y|\mu, \sigma) = \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1} \quad (1.46)$$

for $0 < y < 1$, where $\alpha = \mu(1 - \sigma^2)/\sigma^2$ and $\beta = (1 - \mu)(1 - \sigma^2)/\sigma^2$, $\alpha > 0$, and $\beta > 0$ and hence $0 < \mu < 1$ and $0 < \sigma < 1$. [Note the relationship between parameters (μ, σ) and (α, β) is given by $\mu = \alpha/(\alpha + \beta)$ and $\sigma = (\alpha + \beta + 1)^{-1/2}$.] In this parameterization, the mean of Y is $E(Y) = \mu$ and the variance is $Var(Y) = \sigma^2\mu(1 - \mu)$.

1.8.2 Beta inflated distribution (BEINF, BEINF0, BEINF1)

The beta inflated distribution is appropriate when the response variable takes values in a known restricted range including the endpoints of the range. Appropriate standardization can be applied to make the range of the response variable $[0, 1]$, i.e. from zero to one including the endpoints. Values zero and one for Y have non zero probabilities p_0 and p_1 respectively. The probability (density) function of the inflated beta distribution, denoted by $\mathbf{BEINF}(\mu, \sigma, \nu, \tau)$ is defined by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \begin{cases} p_0 & \text{if } y = 0 \\ (1 - p_0 - p_1) \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1} & \text{if } 0 < y < 1 \\ p_1 & \text{if } y = 1 \end{cases} \quad (1.47)$$

for $0 \leq y \leq 1$, where $\alpha = \mu(1 - \sigma^2)/\sigma^2$, $\beta = (1 - \mu)(1 - \sigma^2)/\sigma^2$, $p_0 = \nu(1 + \nu + \tau)^{-1}$, $p_1 = \tau(1 + \nu + \tau)^{-1}$ so $\alpha > 0$, $\beta > 0$, $0 < p_0 < 1$, $0 < p_1 < 1 - p_0$. Hence $\mathbf{BEINF}(\mu, \sigma, \nu, \tau)$ has parameters $\mu = \alpha/(\alpha + \beta)$ and $\sigma = (\alpha + \beta + 1)^{-1/2}$, $\nu = p_0/p_2$, $\tau = p_1/p_2$ where $p_2 = 1 - p_0 - p_1$. Hence $0 < \mu < 1$, $0 < \sigma < 1$, $\nu > 0$ and $\tau > 0$. Note that $E(y) = \frac{\tau + \mu}{(1 + \nu + \tau)}$.

The probability (density) function of the inflated at zero beta distribution, denoted by $\mathbf{BEINF0}(\mu, \sigma, \nu)$ is defined by

$$f_Y(y|\mu, \sigma, \nu) = \begin{cases} p_0 & \text{if } y = 0 \\ (1 - p_0) \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1} & \text{if } 0 < y < 1 \end{cases} \quad (1.48)$$

for $0 \leq y < 1$, where $\alpha = \mu(1 - \sigma^2)/\sigma^2$, $\beta = (1 - \mu)(1 - \sigma^2)/\sigma^2$, $p_0 = \nu(1 + \nu)^{-1}$, so $\alpha > 0$, $\beta > 0$, $0 < p_0 < 1$. Hence $\mathbf{BEINF0}(\mu, \sigma, \nu)$ has parameters $\mu = \alpha/(\alpha + \beta)$ and $\sigma = (\alpha + \beta + 1)^{-1/2}$, $\nu = p_0/1 - p_0$. Hence $0 < \mu < 1$, $0 < \sigma < 1$, $\nu > 0$. Note that for $\mathbf{BEINF0}(\mu, \sigma, \nu)$, $E(y) = \frac{\mu}{(1 + \nu)}$.

The probability (density) function of the inflated beta distribution, denoted by $\mathbf{BEINF1}(\mu, \sigma, \nu)$ is defined by

$$f_Y(y|\mu, \sigma, \nu) = \begin{cases} (1 - p_1) \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1} & \text{if } 0 < y < 1 \\ p_1 & \text{if } y = 1 \end{cases} \quad (1.49)$$

for $0 < y \leq 1$, where $\alpha = \mu(1 - \sigma^2)/\sigma^2$, $\beta = (1 - \mu)(1 - \sigma^2)/\sigma^2$, $p_1 = \tau(1 + \tau)^{-1}$ so $\alpha > 0$, $\beta > 0$, $0 < p_1 < 1$. Hence **BEINF1**(μ, σ, ν) has parameters $\mu = \alpha/(\alpha + \beta)$ and $\sigma = (\alpha + \beta + 1)^{-1/2}$, $\nu = p_1/(1 - p_2)$. Hence $0 < \mu < 1$, $0 < \sigma < 1$, $\nu > 0$. Note that $E(y) = \frac{\nu + \mu}{(1 + \nu)}$.

For different parametrization of the **BEINF0**(μ, σ, ν) and **BEINF1**(μ, σ, ν) distributions see also **BEZI**(μ, σ, ν) and **BEOI**(μ, σ, ν) distributions contributed to **gamlss** by Raydonal Ospina, Ospina and Ferrari (2010).

1.8.3 Generalized Beta type 1 distribution (GB1)

The generalized beta type 1 distribution is defined by assuming $Z = Y^\tau / [\nu + (1 - \nu)Y^\tau] \sim BE(\mu, \sigma)$. Hence, the pdf of generalized beta type 1 distribution, denoted by **GB1**(μ, σ, ν, τ), is given by

$$f_Y(y|\mu, \sigma, \nu, \tau) = \frac{\tau \nu^\beta y^{\tau\alpha-1} (1 - y^\tau)^{\beta-1}}{B(\alpha, \beta) [\nu + (1 - \nu)y^\tau]^{\alpha+\beta}} \quad (1.50)$$

for $0 < y < 1$, where $\alpha = \mu(1 - \sigma^2)/\sigma^2$ and $\beta = (1 - \mu)(1 - \sigma^2)/\sigma^2$, $\alpha > 0$ and $\beta > 0$. Hence, **GB1**(μ, σ, ν, τ) has adopted parameters $\mu = \alpha/(\alpha + \beta)$, $\sigma = (\alpha + \beta + 1)^{-1/2}$, ν and τ , where $0 < \mu < 1$, $0 < \sigma < 1$, $\nu > 0$ and $\tau > 0$. The beta **BE**(μ, σ) distribution is a special case of **GB1**(μ, σ, ν, τ) where $\nu = 1$ and $\tau = 1$.

1.9 Binomial type data one parameter distributions

1.9.1 The Binomial distribution (BI)

The probability function of the binomial distribution, denoted here as **BI**(n, μ), is given by

$$p_Y(y|n, \mu) = P(Y = y|n, \mu) = \frac{n!}{y!(n - y)!} \mu^y (1 - \mu)^{n-y}$$

for $y = 0, 1, 2, \dots, n$, where $0 < \mu < 1$, (and n is a known positive integer), with $E(Y) = n\mu$ and $Var(Y) = n\mu(1 - \mu)$. See Johnson *et al.* (1993), p 105 where $\mu = p$.

1.10 Binomial type data two parameters distributions

1.10.1 Beta Binomial distribution (BB)

The probability function of the beta binomial distribution denoted here as **BB**(n, μ, σ) is given by

$$p_Y(y|\mu, \sigma) = \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)} \frac{\Gamma(\frac{1}{\sigma})\Gamma(y + \frac{\mu}{\sigma})\Gamma[n + \frac{(1-\mu)}{\sigma} - y]}{\Gamma(n + \frac{1}{\sigma})\Gamma(\frac{\mu}{\sigma})\Gamma(\frac{1-\mu}{\sigma})} \quad (1.51)$$

for $y = 0, 1, 2, \dots, n$, where $0 < \mu < 1$ and $\sigma > 0$ (and n is a known positive integer). Note that $E(Y) = n\mu$ and $Var(Y) = n\mu(1 - \mu) \left[1 + \frac{\sigma}{1+\sigma}(n-1)\right]$.

The binomial **BI**(n, μ) distribution is the limiting distribution of **BB**(n, μ, σ) as $\sigma \rightarrow 0$. For $\mu = 0.5$ and $\sigma = 0.5$, **BB**(n, μ, σ) is a uniform distribution.

1.10.2 Zero altered (or adjusted) binomial (ZABI)

Let $Y = 0$ with probability σ and $Y \sim BIt(n, \mu)$ with probability $(1 - \sigma)$, where $BIt(n, \mu)$ is a Binomial truncated at zero distribution, then Y has a zero altered (or adjusted) binomial distribution, denoted by **ZABI**(n, μ, σ), given by

$$p_Y(y|n, \mu, \sigma) = \begin{cases} \sigma, & \text{if } y = 0 \\ \frac{(1-\sigma)n!\mu^y(1-\mu)^{n-y}}{[1-(1-\mu)^n]y!(n-y)!}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.52)$$

For $0 < \mu < 1$, and $0 < \sigma < 1$. The mean and variance of Y are given by

$$E(Y) = \frac{(1 - \sigma) n \mu}{[1 - (1 - \mu)^n]}$$

and

$$Var(Y) = \frac{n\mu(1-\sigma)(1-\mu+n\mu)}{[1-(1-\mu)^n]} - [E(Y)]^2$$

respectively.

1.10.3 Zero inflated binomial (ZIBI)

Let $Y = 0$ with probability σ and $Y \sim BI(n, \mu)$ with probability $(1 - \sigma)$, then Y has a zero inflated binomial distribution, denoted by **ZIBI**(n, μ, σ), given by

$$p_Y(y|n, \mu, \sigma) = \begin{cases} \sigma + (1 - \sigma)(1 - \mu)^n, & \text{if } y = 0 \\ \frac{(1-\sigma)n!\mu^y(1-\mu)^{n-y}}{y!(n-y)!}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.53)$$

For $0 < \mu < 1$, and $0 < \sigma < 1$. The mean and variance of Y are given by

$$E(Y) = (1 - \sigma) n \mu$$

and

$$Var(Y) = n\mu(1-\sigma)[1-\mu+n\mu\sigma]$$

respectively.

1.11 Binomial type data three parameters distributions

1.11.1 Zero altered (or adjusted) beta binomial (ZABB)

Let $Y = 0$ with probability ν and $Y \sim BBtr(n, \mu, \sigma)$ with probability $(1 - \nu)$, where $BBtr(n, \mu, \sigma)$ is a beta binomial truncated at zero distribution, then Y has a zero altered (or adjusted) beta binomial distribution, denoted by **ZABB**(n, μ, σ, ν), given by

$$p_Y(y|n, \mu, \sigma, \nu) = \begin{cases} \nu, & \text{if } y = 0 \\ \frac{(1-\nu)p_{Y'}(y|n, \mu, \sigma)}{[1-p_{Y'}(0|n, \mu, \sigma)]}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.54)$$

where $Y' \sim BB(n, \mu, \sigma)$. For $0 < \mu < 1$, $\sigma > 0$ and $0 < \nu < 1$. The mean and variance of Y are given by

$$E(Y) = \frac{(1 - \nu) n \mu}{[1 - p_{Y'}(0|n, \mu, \sigma)]}$$

and

$$Var(Y) = \frac{(1 - \nu) \left\{ n\mu(1 - \mu) \left[1 + \frac{\sigma}{1 + \sigma} (n - 1) \right] - n^2 \mu^2 \right\}}{[1 - p_{Y'}(0|n, \mu, \sigma)]} - [E(Y)]^2$$

respectively.

1.12 Binomial type data three parameters distributions

1.12.1 Zero inflated beta binomial (ZIBB)

Let $Y = 0$ with probability ν and $Y \sim BB(n, \mu, \sigma)$ with probability $(1 - \nu)$, then Y has a zero inflated beta binomial distribution, denoted by **ZIBB**(n, μ, σ, ν), given by

$$p_Y(y|n, \mu, \sigma, \nu) = \begin{cases} \nu + (1 - \nu) p_{Y'}(0|n, \mu, \sigma), & \text{if } y = 0 \\ (1 - \nu) p_{Y'}(y|n, \mu, \sigma), & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.55)$$

For $0 < \mu < 1$, $\sigma > 0$ and $0 < \nu < 1$ where $Y' \sim BB(n, \mu, \sigma)$. The mean and variance of Y are given by

$$E(Y) = (1 - \nu) n\mu$$

and

$$Var(Y) = (1 - \nu) n\mu(1 - \mu) \left[1 + \frac{\sigma}{1 + \sigma} (n - 1) \right] + \nu(1 - \nu) n^2 \mu^2$$

respectively.

1.13 Count data one parameter distributions

1.13.1 Poisson distribution (PO)

Poisson distribution

The probability function of the Poisson distribution, denoted here as **PO**(μ), is given by

$$p_Y(y|\mu) = P(Y = y|\mu) = \frac{e^{-\mu} \mu^y}{y!} \quad (1.56)$$

where $y = 0, 1, 2, \dots$, where $\mu > 0$, with $E(Y) = \mu$ and $Var(Y) = \mu$. [See Johnson *et al.* (1993), p 151.] The moment ratios of the distribution are given by $\sqrt{\beta_1} = \mu^{-0.5}$ and $\beta_2 = 3 + \mu^{-1}$ respectively. Note that the Poisson distribution has the property that $E[Y] = Var[Y]$ and that $\beta_2 - \beta_1 - 3 = 0$. The coefficient of variation of the distribution is given by $\mu^{-0.5}$. The index of dispersion, that is, the ratio $Var[Y]/E[Y]$ is equal to one for the Poisson distribution. For $Var[Y] > E[Y]$ we have overdispersion and for $Var[Y] < E[Y]$ we have underdispersion or repulsion. The distribution is skew for small values of μ , but almost symmetric for large μ values.

1.13.2 Logarithmic distribution (LG)

The probability function of the logarithmic distribution, denoted here as **LG**(μ), is given by

$$p_Y(y|\mu) = P(Y = y|\mu) = \frac{\alpha\mu^y}{y}, \quad \text{for } y = 1, 2, \dots \quad (1.57)$$

where $\alpha = -[\log(1 - \mu)]^{-1}$ for $0 < \mu < 1$. Note that the range of Y starts from 1. The mean and variance of Y are given by $E(Y) = \frac{\alpha\mu}{(1-\mu)}$ and $Var(Y) = \frac{\alpha\mu(1-\alpha\mu)}{(1-\mu)^2}$, see Johnson *et al.* (2005) p.302-342.

1.14 Count data two parameters distributions

1.14.1 Negative Binomial distribution (NBI, NBII)

First parameterization: Negative Binomial type I (NBI)

The probability function of the negative binomial distribution type I, denoted here as **NBI**(μ, σ), is given by

$$p_Y(y|\mu, \sigma) = \frac{\Gamma(y + \frac{1}{\sigma})}{\Gamma(\frac{1}{\sigma})\Gamma(y+1)} \left(\frac{\sigma\mu}{1 + \sigma\mu} \right)^y \left(\frac{1}{1 + \sigma\mu} \right)^{1/\sigma}$$

for $y = 0, 1, 2, \dots$, where $\mu > 0$, $\sigma > 0$ with $E(Y) = \mu$ and $Var(Y) = \mu + \sigma\mu^2$. [This parameterization is equivalent to that used by Anscombe (1950) except he used $\alpha = 1/\sigma$, as pointed out by Johnson *et al.* (1993), p 200, line 5.]

Second parameterization: Negative Binomial type II (NBII)

The probability function of the negative binomial distribution type II, denoted here as **NBII**(μ, σ), is given by

$$p_Y(y|\mu, \sigma) = \frac{\Gamma(y + \mu/\sigma)\sigma^y}{\Gamma(\mu/\sigma)\Gamma(y+1)(1 + \sigma)^{y+\mu/\sigma}}$$

for $y = 0, 1, 2, \dots$, where $\mu > 0$ and $\sigma > 0$. Note $E(Y) = \mu$ and $Var(Y) = (1 + \sigma)\mu$, so σ is a dispersion parameter [This parameterization was used by Evans (1953) as pointed out by Johnson *et al.* (1993) p 200 line 7.]

1.14.2 Poisson-inverse Gaussian distribution (PIG)

The probability function of the Poisson-inverse Gaussian distribution, denoted by **PIG**(μ, σ), is given by

$$p_Y(y|\mu, \sigma) = \left(\frac{2\alpha}{\pi} \right)^{\frac{1}{2}} \frac{\mu^y e^{1/\sigma} K_{y-\frac{1}{2}}(\alpha)}{(\alpha\sigma)^y y!}$$

where $\alpha^2 = \frac{1}{\sigma^2} + \frac{2\mu}{\sigma}$, for $y = 0, 1, 2, \dots, \infty$ where $\mu > 0$ and $\sigma > 0$ and $K_\lambda(t) = \frac{1}{2} \int_0^\infty x^{\lambda-1} \exp\{-\frac{1}{2}t(x+x^{-1})\}dx$ is the modified Bessel function of the third kind. [Note that the above parameterization was used by Dean, Lawless and Willmot (1989). It is also a special case of the **gamlss.family** distribution **SI**(μ, σ, ν) when $\nu = -\frac{1}{2}$.]

1.14.3 Zero inflated poisson (ZIP, ZIP2)

First parameterization (ZIP)

Let $Y = 0$ with probability σ and $Y \sim Po(\mu)$ with probability $(1 - \sigma)$, then Y has a zero inflated Poisson distribution, denoted by **ZIP**(μ, σ), given by

$$p_Y(y|\mu, \sigma) = \begin{cases} \sigma + (1 - \sigma)e^{-\mu}, & \text{if } y = 0 \\ (1 - \sigma)\frac{\mu^y}{y!}e^{-\mu}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.58)$$

See Johnson *et al* (1993), p 186, equation (4.100) for this parametrization. This parametrization was also used by Lambert (1992). The mean of Y in this parametrization is given by $E(Y) = (1 - \sigma)\mu$ and its variance by $Var(Y) = \mu(1 - \sigma)[1 + \mu\sigma]$.

Second parameterization (ZIP2)

A different parameterization of the zero inflated poisson distribution, denoted by **ZIP2**(μ, σ), is given by

$$p_Y(y|\mu, \sigma) = \begin{cases} \sigma + (1 - \sigma)e^{-\left(\frac{\mu}{1-\sigma}\right)}, & \text{if } y = 0 \\ (1 - \sigma)\frac{\mu^y}{y!(1-\sigma)^y}e^{-\left(\frac{\mu}{1-\sigma}\right)}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.59)$$

The mean of Y in (1.59) is given by $E(Y) = \mu$ and the variance by $Var(Y) = \mu + \mu^2 \frac{\sigma}{(1-\sigma)}$.

1.14.4 Zero altered (or adjusted) poisson (ZAP)

Let $Y = 0$ with probability σ and $Y \sim POtr(\mu)$ with probability $(1 - \sigma)$, where $POtr(\mu)$ is a Poisson truncated at zero distribution, then Y has a zero adjusted Poisson distribution, denoted by **ZAP**(μ, σ), given by

$$p_Y(y|\mu, \sigma) = \begin{cases} \sigma, & \text{if } y = 0 \\ \frac{(1-\sigma)e^{-\mu}\mu^y}{y!(1-e^{-\mu})}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.60)$$

The mean of Y in this parametrization is given by $E(Y) = (1 - \sigma)\mu / (1 - e^{-\mu})$ and its variance by $Var(Y) = \frac{(1-\sigma)}{(1-e^{-\mu})} [\mu + \mu^2] - [E(Y)]^2$.

1.14.5 Zero altered (or adjusted) logarithmic (ZALG)

Let $Y = 0$ with probability σ and $Y \sim LG(\mu)$, a logarithmic distribution with probability $(1 - \sigma)$, then Y has a zero altered (adjusted) logarithmic distribution, denoted by **ZALG**(μ, σ), with probability function given by

$$p_Y(y|\mu, \sigma) = \begin{cases} \sigma, & \text{if } y = 0 \\ (1 - \sigma)\frac{\alpha\mu^y}{y}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.61)$$

where $\alpha = -[\log(1 - \mu)]^{-1}$ for $0 < \mu < 1$ and $0 < \sigma < 1$. The mean and variance of Y are given by $E(Y) = \frac{(1-\sigma)\alpha\mu}{(1-\mu)}$ and its variance by $Var(Y) = \frac{(1-\sigma)\alpha\mu[1-(1-\sigma)\alpha\mu]}{(1-\mu)^2}$.

1.15 Count data three parameters distributions

1.15.1 Delaporte distribution (DEL)

The probability function of the Delaporte distribution, denoted by **DEL**(μ, σ, ν), is given by

$$p_Y(y|\mu, \sigma, \nu) = \frac{e^{-\mu\nu}}{\Gamma(1/\sigma)} [1 + \mu\sigma(1 - \nu)]^{-1/\sigma} S \quad (1.62)$$

where

$$S = \sum_{j=0}^y \binom{y}{j} \frac{\mu^y \nu^{y-j}}{y!} \left[\mu + \frac{1}{\sigma(1 - \nu)} \right]^{-j} \Gamma\left(\frac{1}{\sigma} + j\right)$$

for $y = 0, 1, 2, \dots, \infty$ where $\mu > 0$, $\sigma > 0$ and $0 < \nu < 1$. This distribution is a reparameterization of the distribution given by Wimmer and Altman (1999) p 515-516 where $\alpha = \mu\nu$, $k = 1/\sigma$ and $\rho = [1 + \mu\sigma(1 - \nu)]^{-1}$. The mean of Y is given by $E(Y) = \mu$ and the variance by $Var(Y) = \mu + \mu^2\sigma(1 - \nu)^2$.

1.15.2 Sichel distribution (SI, SICHEL)

First parameterization (SI)

The probability function of the first parameterization of the Sichel distribution, denoted by **SI**(μ, σ, ν), is given by

$$p_Y(y|\mu, \sigma, \nu) = \frac{\mu^y K_{y+\nu}(\alpha)}{(\alpha\sigma)^{y+\nu} y! K_\nu(\frac{1}{\sigma})} \quad (1.63)$$

where $\alpha^2 = \frac{1}{\sigma^2} + \frac{2\mu}{\sigma}$, for $y = 0, 1, 2, \dots, \infty$ where $\mu > 0$, $\sigma > 0$ and $-\infty < \nu < \infty$ and $K_\lambda(t) = \frac{1}{2} \int_0^\infty x^{\lambda-1} \exp\{-\frac{1}{2}t(x+x^{-1})\} dx$ is the modified Bessel function of the third kind. Note that the above parameterization is different from Stein, Zucchini and Juritz (1988) who use the above probability function but treat μ , α and ν as the parameters. Note that $\sigma = [(\mu^2 + \alpha^2)^{\frac{1}{2}} - \mu]^{-1}$.

Second parameterization (SICHEL)

The second parameterization of the Sichel distribution, Rigby, Stasinopoulos and Akantziliotou (2008), denoted by **SICHEL**(μ, σ, ν), is given by

$$p_Y(y|\mu, \sigma, \nu) = \frac{(\mu/c)^y K_{y+\nu}(\alpha)}{y! (\alpha\sigma)^{y+\nu} K_\nu(\frac{1}{\sigma})} \quad (1.64)$$

for $y = 0, 1, 2, \dots, \infty$, where $\alpha^2 = \sigma^{-2} + 2\mu(c\sigma)^{-1}$. The mean of Y is given by $E(Y) = \mu$ and the variance by $Var(Y) = \mu + \mu^2 [2\sigma(\nu + 1)/c + 1/c^2 - 1]$.

1.15.3 Zero inflated negative binomial distribution (ZINBI)

Let $Y = 0$ with probability ν and $Y \sim NBI(\mu, \sigma)$, with probability $(1 - \nu)$, then Y has a zero inflated negative binomial distribution, denoted by **ZINBI**(μ, σ, ν), with probability function given by

$$p_Y(y|\mu, \sigma, \nu) = \begin{cases} \nu + (1 - \nu) p_{Y'}(0|\mu, \sigma), & \text{if } y = 0 \\ (1 - \nu) p_{Y'}(y|\mu, \sigma), & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.65)$$

for $\mu > 0$, $\sigma > 0$ and $0 < \nu < 1$, where $Y' \sim NBI(\mu, \sigma)$ so

$$p_{Y'}(0|\mu, \sigma) = (1 + \sigma\mu)^{-\frac{1}{\sigma}}$$

and

$$p_{Y'}(y|\mu, \sigma) = \frac{\Gamma(y + \frac{1}{\sigma})}{\Gamma(\frac{1}{\sigma})\Gamma(y+1)} \left(\frac{\sigma\mu}{1 + \sigma\mu} \right)^y \left(\frac{1}{1 + \sigma\mu} \right)^{1/\sigma}$$

for $y = 0, 1, 2, 3, \dots$. The mean of Y is given by $E(Y) = (1 - \nu)\mu$ and the variance by $Var(Y) = \mu(1 - \nu)[1 + (\sigma + \nu)\mu]$, since for any three parameter zero inflated distribution

$$E(Y) = (1 - \nu)E(Y')$$

and

$$Var(Y) = (1 - \nu)Var(Y') + \nu(1 - \nu)[E(Y')]^2.$$

1.15.4 Zero altered (or adjusted) negative binomial distribution (ZANBI)

Let $Y = 0$ with probability ν and $Y \sim NBIt(\mu, \sigma)$, with probability $(1 - \nu)$, where $NBIt(\mu, \sigma)$ is a negative binomial truncated at zero distribution, then Y has a zero altered (or adjusted) negative binomial distribution, denoted by **ZANBI**(μ, σ, ν), with probability function given by

$$p_Y(y|\mu, \sigma, \nu) = \begin{cases} \nu, & \text{if } y = 0 \\ \frac{(1-\nu)p_{Y'}(y|\mu, \sigma)}{[1 - p_{Y'}(0|\mu, \sigma)]}, & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.66)$$

for $\mu > 0$, $\sigma > 0$ and $0 < \nu < 1$ where $Y' \sim NBI(\mu, \sigma)$ so

$$p_{Y'}(0|\mu, \sigma) = (1 + \sigma\mu)^{-\frac{1}{\sigma}}$$

. and

$$p_{Y'}(y|\mu, \sigma) = \frac{\Gamma(y + \frac{1}{\sigma})}{\Gamma(\frac{1}{\sigma})\Gamma(y+1)} \left(\frac{\sigma\mu}{1 + \sigma\mu} \right)^y \left(\frac{1}{1 + \sigma\mu} \right)^{1/\sigma}$$

for $y = 0, 1, 2, 3, \dots$. The mean of Y is given by

$$E(Y) = \frac{(1 - \nu)\mu}{[1 - p_{Y'}(0|\mu, \sigma)]}$$

and the variance by

$$Var(Y) = \frac{(1 - \nu)}{[1 - p_{Y'}(0|\mu, \sigma)]} \{ \mu + (\sigma + 1)\mu^2 \} - [E(Y)]^2$$

since for any three parameter zero altered distribution we have

$$E(Y) = \frac{(1 - \nu)E(Y')}{[1 - p_{Y'}(0|\mu, \sigma)]}$$

and

$$Var(Y) = \frac{(1 - \nu)}{[1 - p_{Y'}(0|\mu, \sigma)]} \{ Var(Y') + [E(Y')]^2 \} - [E(Y)]^2.$$

1.15.5 Zero inflated Poisson inverse Gaussian distribution (ZIPIG)

Let $Y = 0$ with probability ν and $Y \sim PIG(\mu, \sigma)$, with probability $(1 - \nu)$, then Y has a zero inflated Poisson inverse Gaussian distribution, denoted by **ZIPIG** (μ, σ, ν) , with probability function given by

$$p_Y(y|\mu, \sigma, \nu) = \begin{cases} \nu + (1 - \nu) p_{Y'}(0|\mu, \sigma), & \text{if } y = 0 \\ (1 - \nu) p_{Y'}(y|\mu, \sigma), & \text{if } y = 1, 2, 3, \dots \end{cases} \quad (1.67)$$

for $\mu > 0$, $\sigma > 0$ and $0 < \nu < 1$, where $Y' \sim PIG(\mu, \sigma)$. The mean of Y is given by $E(Y) = (1 - \nu)\mu$ and the variance by $Var(Y) = \mu(1 - \nu)[1 + (\sigma + \nu)\mu]$.

Bibliography

- [1] **Anscombe, F. J.** (1950). Sampling theory of the negative binomial and logarithmic series approximations. *Biometrika*, **37**: 358–382.
- [2] **Azzalini, A.** (1986). Further results on a class of distributions which includes the normal ones. *Statistica*, **46**: 199:208.
- [3] **Azzalini, A. and Capitanio, A.** (1985). A class of distributions which includes the normal ones. *Scand. J. Statist.*, **12**: 171–178.
- [4] **Azzalini, A. and Capitanio, A.** (2003). Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t -distribution. *J. R. Statist. Soc. B*, **65**: 367–389.
- [5] **Box, G. E. P. and Cox, D. R.** (1964). An analysis of transformations (with discussion). *J. R. Statist. Soc. B.*, **26**: 211–252.
- [6] **Cole, T. J. and Green, P. J.** (1992). Smoothing reference centile curves: the LMS method and penalized likelihood. *Statist. Med.*, **11**: 1305–1319.
- [7] **Crowder, M. J., Kimber, A. C., Smith R. L. and Sweeting, T. J.** (1991). *Statistical Analysis of Reliability Data*. Chapman and Hall, London.
- [8] **Dean, C., Lawless, J. F. and Willmot, G. E.** (1989). A mixed Poisson-inverse-Gaussian regression model. *Canadian Journal of Statistics*, **17**: 171–181.
- [9] **DiCiccio, T. J. and Monti, A. C.** (2004). Inferential Aspects of the Skew Exponential Power Distribution. *J. Am. Statist. Ass.*, **99**: 439–450.
- [10] **Evans, D. A.** (1953). Experimental evidence concerning contagious distributions in ecology. *Biometrika*, **40**: 186–211.
- [11] **Fernandez, C. and Steel, M. F. J.** (1998). On Bayesian Modelling of Fat Tails and Skewness. *J. Am. Statist. Ass.*, **93**: 359–371.
- [12] **Fernandez, C., Osiewalski, J. and Steel, M. J. F.** (1995). Modeling and inference with v-spherical distributions. *J. Am. Statist. Ass.*, **90**: 1331–1340.
- [13] **Johnson, N. L.** (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, **36**: 149–176.
- [14] **Johnson, N. L., Kotz, S. and Balakrishnan, N.** (1994). *Continuous Univariate Distributions, Volume I, 2nd edn.* Wiley, New York.

- [15] **Johnson, N. L., Kotz, S. and Balakrishnan, N.** (1995). *Continuous Univariate Distributions, Volume II, 2nd edn.* Wiley, New York.
- [16] **Johnson, N. L., Kotz, S. and Kemp, A. W.** (2005). *Univariate Discrete Distributions, 3rd edn.* Wiley, New York.
- [17] **Jones, M. C.** (2005). In discussion of Rigby, R. A. and Stasinopoulos, D. M. (2005) Generalized additive models for location, scale and shape,. *Applied Statistics*, **54**: 507–554.
- [18] **Jones, M. C. and Faddy, M. J.** (2003). A skew extension of the t distribution, with applications. *J. Roy. Statist. Soc B*, **65**: 159–174.
- [19] **Jørgensen, B.** (1982). *Statistical Properties of the Generalized Inverse Gaussian Distribution, Lecture Notes in Statistics No.9.* Springer-Verlag, New York.
- [20] **Lambert, D.** (1992). Zero-inflated Poisson Regression with an application to defects in Manufacturing. *Technometrics*, **34**: 1–14.
- [21] **Lopatatzidis, A. and Green, P. J.** (2000). Nonparametric quantile regression using the gamma distribution. *submitted for publication*.
- [22] **McDonald, J. B.** (1991). Parametric models for partially adaptive estimation with skewed and leptokurtic residuals. *Economic Letters*, **37**: 273–278.
- [23] **McDonald, J. B.** (1996). Probability Distributions for Financial Models. In: Maddala, G. S. and Rao, C. R. (eds.), *Handbook of Statistics, Vol. 14*, pp. 427–460. Elsevier Science.
- [24] **McDonald, J. B. and Newey, W. K.** (1988). Partially adaptive estimation of regression models via the generalized t distribution. *Econometric Theory*, **4**: 428–457.
- [25] **McDonald, J. B. and Xu, Y. J.** (1995). A generalisation of the beta distribution with applications. *Journal of Econometrics*, **66**: 133–152.
- [26] **Nelson, D. B.** (1991). Conditional heteroskedasticity in asset returns: a new approach. *Econometrica*, **59**: 347–370.
- [27] **R., O. and P., F. S. L.** (2010). Inflated beta distributions. *Statistical Papers*, **23**: 111–126.
- [28] **Rigby, R. A. and Stasinopoulos, D. M.** (1994). Robust fitting of an additive model for variance heterogeneity. In: Dutter, R. and Grossmann, W. (eds.), *COMPSTAT : Proceedings in Computational Statistics*, pp. 263–268. Physica, Heidelberg.
- [29] **Rigby, R. A. and Stasinopoulos, D. M.** (2004). Smooth centile curves for skew and kurtotic data modelled using the Box-Cox Power Exponential distribution. *Statistics in Medicine*, **23**: 3053–3076.
- [30] **Rigby, R. A. and Stasinopoulos, D. M.** (2006). Using the Box-Cox t distribution in GAMLSS to model skewness and kurtosis. *Statistical Modelling*, **6**: 209–229.
- [31] **Rigby, R. A. Stasinopoulos, D. M. and K., A.** (2008). A framework for modelling overdispersed count data, including the Poisson-shifted generalized inverse Gaussian distribution. *Computational Statistics and Data analysis*, **n print**.

- [32] **Stein, G. Z., Zucchini, W. and Juritz, J. M.** (1987). Parameter Estimation of the Sichel Distribution and its Multivariate Extension. *Journal of American Statistical Association*, **82**: 938–944.
- [33] **Subbotin, M. T.** (1923). On the law of frequency of errors. *Mathematicheskii Sbornik*, **31**: 296–301.
- [34] **Wimmer, G. and Altmann, G.** (1999). *Thesaurus of univariate discrete probability distributions*. Stamm Verlag, Essen, Germany.

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